Factorizable S-matrices from nonlocal Z_N charges

1992 J. Phys. A: Math. Gen. 25 2693

(http://iopscience.iop.org/0305-4470/25/9/037)

J. Phys. A: Math. Gen. **25** (1992) 2693-2709. Printed in the UK

Factorizable S-matrices from non-local Z_N charges

H J de Vega† and V A Fateev‡†

† Laboratoire de Physique Théorique et Hautes Energies§, Université Paris VI, Tour 16, 1[er] ét. 4, Place Jussieu, Paris Cedex 05, France

‡ Landau Institute for Theoretical Physics, Academy of Sciences of the USSR, ul. Kossygina 2, 117334 Moscow, Russia

Received 23 October 1991

Abstract. Conserved non-local Z_N charges Q and $\bar{Q}$ are defined through their action on asymptotic particle states. This action entails a non-trivial coproduct. S-matrices invariant under these Z_N charges are explicitly found. They enjoy in addition P, C, T, crossing invariance and unitarity. The unitarization factors are calculated. Finally, we identify perturbed conformal field theories described by these factorized S-matrices.

1. Introduction

Perturbing conformal field theories (CFT) by relevant operators, one finds as a rule, massive quantum field theories (QFT). These QFT contain information about the scaling limit in the vicinity of the fixed point and in many important cases turn out to be integrable theories possessing factorizable S-matrices [1, 2].

Integrable QFT are characterized by the presence of conserved charges in an infinite number. These charges may be local or non-local in the basic fields. Local charges have integer (Lorentz) spin whereas non-local charges have often a non-integer spin.

The conservation of a non-local charge Q imposes strong constraints on the S-matrix. That is, the requirement that Q has the same form on ingoing and outgoing particle states usually determines the S-matrix up to some free parameters (and CDD poles) [3]. In addition the whole set of non-local conserved charges constitute an infinite dimensional non-Abelian symmetry of Yang–Baxter type in the QFT [4]. In particular one finds the quantum group invariance as a limiting case [5].

The action of non-local charges on multiparticle states usually follows non-trivial coproduct rules [3, 7]. In perturbed CFT, the study of non-local charges with fractional spin starts in [8] with the perturbed tricritical Ising and Z_3-Potts models.

Non-local conserved charges appear in many different theories [3-7]. For example, the B_n-Toda QFT possess a pair of fermionic charges $Q_\pm$ with spin$\pm(n-\frac{1}{2})$. These anticommuting charges ($Q_+Q_-=-Q_-Q_+$) completely determine the S-matrix of the model as shown in [7]. (The parameter left free is a function of the coupling constant.)

The purpose of this paper is to investigate Z_N-conserved charges. We define them through

$$
\begin{aligned}
&Q\bar{Q}=\omega^s\bar{Q}Q \\
&Q^N=P_s \qquad \bar{Q}^N=\bar{P}_s
\end{aligned}
\tag{1.1}
$$

§ Laboratoire Associé au CNRS UA 280.

where P_s, $\bar{P}_s$ are Z_N-invariant local integrals of motion with spin $\pm s$ and $\omega = e^{2\pi i/N}$. The charges considered in [7] correspond to $N=2$. Here we shall consider N odd, $N \geqslant 3$.

We proceed as follows. Our one-particle states are of the form $|\sigma, \theta\rangle$ where $\sigma = 0, 1, \ldots, N-1 \bmod N$ and θ is the particle rapidity. The action of Q and $\bar{Q}$ on them is as follows

$$\begin{aligned} Q|\sigma, \theta\rangle &= \lambda_\theta \omega^{sd\sigma}|\sigma+1, \theta\rangle \\ \bar{Q}|\sigma, \theta\rangle &= \bar{\lambda}_\theta \omega^{-s(d+1)\sigma}|\sigma-1, \theta\rangle \end{aligned} \tag{1.2}$$

where $\lambda_\theta = m^{1/N} e^{s\theta/N}$, $\bar{\lambda}_\theta = m^{1/N} e^{-s\theta/N}$ and d an integer to be chosen later as $d = N-1/2$. On two-particle states Q and $\bar{Q}$ act following a non-trivial coproduct defined by equations (3.3). This coproduct and its n-particle generalization provides a representation of the Z_N-charges (1.1) on asymptotic particle states. For even s, we introduce an extra quantum number $\varepsilon = \pm 1$ to allow P_s and $\bar{P}_s$ to be odd under charge conjugation.

We find a Z_N-invariant S-matrix with $s=1$ requiring that Q and $\bar{Q}$ to commute with the S-matrix. We also impose P, T, C and crossing invariance. The derivation of the S-matrix is given in section 4. Our solution can be written as

$$S_{\sigma_3\sigma_4}^{\sigma_1\sigma_2}(\theta) = \delta_{\sigma_1+\sigma_2, \sigma_3+\sigma_4} \phi(\sigma_1+\sigma_2, \sigma_1-\sigma_3 \,|\, \theta) \tag{1.3}$$

where

$$\phi(\sigma, K, \theta) = \omega^{d\sigma K} F_N(\theta) \sum_{m=0}^{N-1} \omega^{\sigma m} X_m(\theta) X_{K-m}(\theta) \tag{1.4}$$

$d = N-1/2$ and $F_N(\theta)$ is a unitarizing factor. In addition; we show in section 4 that this S-matrix is real analytic. S-matrices with $s \neq 1$ are obtained from the $s=1$ solution (1.3)–(1.4) by rescaling $\theta \to s\theta$, $\omega \to \omega^s$ (N not divisible by s, $1 \leqslant s \leqslant N-1$). In addition, for even s we introduce an extra quantum number $\varepsilon = \pm$. Particularly interesting cases are $s = N-1$ (model II) and $s = N-2$ (model I). We investigate that in some detail. The unitarizing factor for model I, $F_N(\theta)$, is explicitly found. It is given by an infinite product of gamma functions (equations (3.46), (3.50) and (3.52)). For model II, an extra quantum number $\varepsilon = \pm 1$ is introduced to characterize the particles. The S-matrix is given explicitly in section 3 ((3.56)–(3.66)). We conjecture that the full mass spectrum of model I(II) coincides with the $B_{(N-1)/2}(D_N)$ mass spectrum. Finally we investigate the perturbed CFT described by the QFT models I and II. Both models turn out to be pairs of parafermionic CFT [8] perturbed by the operator

$$\lambda \int d^2x\, \varepsilon_1 \varepsilon_1'. \tag{1.5}$$

The parafermionic CFT is a Z_N (Z_{2N}) model in the case I(II) with central charge $C_N = 2(N-1)/(N+2)(C_{2N} = (2N-1)/N+1)$. ε_1 and ε_1' are neutral fields (thermal operators) with conformal dimensions

$$D_1 = \bar{D}_1 = \frac{2}{N+2} \,(\text{model I}) \qquad D_1 = \bar{D}_1 = \frac{1}{N+1} \,(\text{model II})$$

associated to each of the parafermionic CDT.

Moreover, we identify the Z_N-charges with the operator

$$Q = \int \psi_1 \psi_1' \, dz + \lambda c \int \Phi_{[2,0]}^{(2)} \Phi_{[2,0]}^{(2)\prime} \, dz \tag{1.6}$$

where ψ_1 and ψ_1' are the parafermionic fields associated to each of the CFT, the fields $\Phi^{(2)}_{[2,0]}$ are defined in [9] and c is a numerical constant. Equations (1.5) and (1.6) hold in models *I* and *II*.

2. Z_3 Non-local charges and their associated S-matrix

We consider two conserved non-local Z_3 charges Q and $\bar{Q}$ with Lorentz spin-$\frac{1}{3}$ acting on asymptotic particle states and satisfying the following relations

$$Q^3 = P_s \qquad \bar{Q}^3 = \bar{P}_s \qquad Q\bar{Q} = \omega\bar{Q}Q \tag{2.1}$$

where $\omega = e^{2\pi i/3}$ and P_s, $\bar{P}_s$ are Z_3-invariant local integrals of motion with spins s. They act as follows on one particle states $|\sigma, \theta\rangle$ where $\sigma = -1, 0, +1 \pmod 3$ and θ in the particle rapidity,

$$\begin{aligned} Q|\sigma, \theta\rangle &= \lambda_\theta \omega^\sigma |\sigma+1, \theta\rangle \\ \bar{Q}|\sigma, \theta\rangle &= \bar{\lambda}_\theta \omega^\sigma |\sigma-1, \theta\rangle. \end{aligned} \tag{2.2}$$

We assume the particle mass to be independent of σ.

Equations (2.1) hold if we choose, for example

$$\lambda_\theta = (m\, e^\theta)^{1/3} \qquad \bar{\lambda}_\theta = (m\, e^{-\theta})^{1/3}. \tag{2.3}$$

It means that we choose Q and $\bar{Q}$ to have (Lorentz) spin $s = \frac{1}{3}$ and $-\frac{1}{3}$, respectively.

The action of Q and $\bar{Q}$ on two (or more) particle states is more subtle. There is a non-trivial coproduct rule that physically reflects the non-local character of these charges. We define

$$Q|\sigma_1, \theta_1; \sigma_2, \theta_2\rangle = \lambda_1 \omega^{\sigma_1} |\sigma_1+1, \theta_1; \sigma_2, \theta_2\rangle + \lambda_2 \omega^{-\sigma_1+\sigma_2} |\sigma_1, \theta_1; \sigma_2+1, \theta_2\rangle \tag{2.4}$$

$$\bar{Q}|\sigma_1, \theta_1; \sigma_2, \theta_2\rangle = \bar{\lambda}_1 \omega^{\sigma_1} |\sigma_1-1, \theta_1; \sigma_2, \theta_2\rangle + \bar{\lambda}_2 \omega^{\sigma_2-\sigma_1} |\sigma_1, \theta_1; \sigma_2-1, \theta_2\rangle. \tag{2.5}$$

Here $\lambda_{1,2} \equiv \lambda_{\theta_{1,2}}$, $\bar{\lambda}_{1,2} \equiv \bar{\lambda}_{\theta_{1,2}}$. The phases ω^{σ_1} and ω^{σ_2} correspond to the action of Q and $\bar{Q}$ on the particles 1 and 2 respectively. The phase $\omega^{-\sigma_1}$ in the second terms accounts for the non-trivial character of the coproduct. It is easy to check that (2.4)–(2.5) guarantee the validity of (2.1) on the two particle space.

The action of Q and $\bar{Q}$ on n-particle states (generalizing (2.4)–(2.5)) is as follows

$$\begin{aligned} Q|\underline{\theta}, \underline{\sigma}\rangle &= \sum_{i=1}^{n} \lambda_i \omega^{\sigma_i - \sum_{j=1}^{i-1}\sigma_j} |\underline{\theta}, \underline{\sigma}^{(i)}\rangle \\ \bar{Q}|\underline{\theta}, \underline{\sigma}\rangle &= \sum_{i=1}^{n} \bar{\lambda}_i \omega^{\sigma_i - \sum_{j=1}^{i-1}\sigma_j} |\underline{\theta}, \bar{\underline{\sigma}}^{(i)}\rangle \end{aligned} \tag{2.6}$$

where $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\underline{\theta} = (\theta_1, \ldots, \theta_n)$, $\underline{\sigma}^{(i)} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i+1, \sigma_{i+1}, \ldots, \sigma_n)$, and $\bar{\underline{\sigma}}^{(i)} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i - 1, \sigma_{i+1}, \ldots, \sigma_n)$.

Let us now consider the S-matrix for the scattering of these particles. To begin with, the two-particle S-matrix

$$S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta) \tag{2.7}$$

where θ is the relative rapidity, vanishes unless $\sigma_1 + \sigma_2 = \sigma_3 + \sigma_4 \pmod 3$. Hence, we set

$$S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta) = \delta_{\sigma_1+\sigma_2, \sigma_3+\sigma_4}\, \alpha(\sigma_1, \sigma_2, \sigma_2 - \sigma_3 | \theta). \tag{2.8}$$

Let us recall how PT, C, P and crossing invariance constrain the S-matrix:

$$PT\colon S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta)=S^{\sigma_3\sigma_4}_{\sigma_1\sigma_2}(\theta)\Rightarrow\alpha(\sigma_1,\sigma_2,K\,|\,\theta)=\alpha(\sigma_1+K,\sigma_2-K,-K\,|\,\theta)\tag{2.9}$$

$$C\colon S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta)=S^{-\sigma_1,-\sigma_2}_{-\sigma_3,-\sigma_4}(\theta)\Rightarrow\alpha(\sigma_1,\sigma_2,K\,|\,\theta)=\alpha(-\sigma_1,-\sigma_2,-K\,|\,\theta)\tag{2.10}$$

$$P\colon S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta)=S^{\sigma_2\sigma_1}_{\sigma_4\sigma_3}(\theta)\Rightarrow\alpha(\sigma_1,\sigma_2,K\,|\,\theta)=\alpha(\sigma_2,\sigma_1,-K\,|\,\theta)\tag{2.11}$$

crossing:

$$S^{\sigma_1\sigma_2}_{\sigma_3\sigma_4}(\theta)=S^{-\sigma_3,\sigma_2}_{-\sigma_1,\sigma_4}(\mathrm{i}\pi-\theta)\Rightarrow\alpha(\sigma_1,\sigma_2,K\,|\,\theta)=\alpha(K-\sigma_2,\sigma_2,\sigma_1+\sigma_2\,|\,\mathrm{i}\pi-\theta).\tag{2.12}$$

Imposing Q and $\bar{Q}$ conservation yields further constraints on the S-matrix elements $\alpha(\sigma_1,\sigma_2,K\,|\,\theta)$. That is, we require Q and $\bar{Q}$ to have an identical form on both in and out states. We find in this way

$$\lambda^2[\alpha(\sigma_1,\sigma_2,K\,|\,\theta)-\omega^{\sigma_1+\sigma_2-K}\alpha(\sigma_1,\sigma_2+1,K+1\,|\,\theta)]$$
$$=\omega^{-K}\alpha(\sigma_1+1,\sigma_2,K\,|\,\theta)-\omega^{\sigma_1+\sigma_2+1}\alpha(\sigma_1,\sigma_2,K+1\,|\,\theta)\tag{2.13}$$

$$\lambda^2[\omega^{\sigma_1}\alpha(\sigma_1-1,\sigma_2,K+1\,|\,\theta)-\omega^{K+K\sigma_2-\sigma_1}\alpha(\sigma_1,\sigma_2,K\,|\,\theta)]$$
$$=\omega^{\sigma_1+K+1}\alpha(\sigma_1,\sigma_2,K+1\,|\,\theta)-\omega^{\sigma_2-\sigma_1}\alpha(\sigma_1,\sigma_2-1,K\,|\,\theta)\tag{2.14}$$

where $\lambda\equiv\sqrt{\lambda_2/\lambda_1}=\mathrm{e}^{-\theta/6}$ and $\theta\equiv\theta_1-\theta_2$ is the relative rapidity.

Equations (2.13)-(2.14) have a solution of the form

$$\alpha(\sigma_1,\sigma_2,K\,|\,\theta)=\phi(\sigma_1+\sigma_2,K\,|\,\theta).\tag{2.15}$$

Equations (2.13)-(2.14) take then the following form

$$\phi(\sigma,K\,|\,\theta)-\omega^{\sigma-K}\phi(\sigma-1,K-1\,|\,\theta)$$
$$=\lambda^2[\omega^{-K}\phi(\sigma-1,K\,|\,\theta)-\omega^{\sigma-1}\phi(\sigma,K-1\,|\,\theta)]\tag{2.16a}$$

$$\phi(\sigma,K\,|\,\theta)-\omega^{\sigma+K}\phi(\sigma-1,K+1\,|\,\theta)$$
$$=\lambda^2[\omega^{K}\phi(\sigma-1,K\,|\,\theta)-\omega^{\sigma-1}\phi(\sigma,K+1\,|\,\theta)].\tag{2.16b}$$

Equation (2.16b) follows from (2.16a) through invariance (2.9). Therefore, it is enough to solve (2.16a). We do that by discrete Fourier transform

$$\phi(\sigma,K)=\omega^{\sigma K}\sum_{\alpha=0,\pm}\omega^{\sigma\alpha}\psi(\alpha,K).\tag{2.17}$$

We find from (2.16a) and (2.17) the simple recursion relation

$$[1-\lambda^2\omega^{K-\alpha}]\psi(\alpha,K)=[\omega^{K+1-\alpha}-\omega^2\lambda^2]\psi(\alpha,K-1).\tag{2.18}$$

Equation (2.18) allows $\psi(\alpha,+)$ and $\psi(\alpha,-)$ to be expressed in terms of $\psi(\alpha,0)$ for all $\alpha=0,\pm$. Furthermore P-invariance (2.11) requires

$$\psi(\alpha,K)=\psi(\alpha-K,-K).\tag{2.19}$$

That is

$$\psi(0,-)=\psi(+,+),\ \psi(0,+)=\psi(-,-)\quad\text{and}\quad\psi(+,0)=\psi(-,0).\tag{2.20}$$

This completely fixes the solution $\psi(\alpha, K)$ up to an overall multiplicative factor $F(\theta)$. After some calculation, we find from equations (2.18)-(2.20)

$$\begin{aligned}
\psi(0,\pm)=\psi(\pm,\pm)&=\frac{F(\theta)}{\sqrt{3}}\left[\cosh\left(\frac{\theta+2\pi \mathrm{i}}{3}\right)+\frac{1}{2}\right]\\
\psi(\mp,\pm)=\psi(\pm,0)&=\frac{F(\theta)}{\sqrt{3}}\left[\cosh\left(\frac{\theta}{3}\right)-1\right]\\
\psi(0,0)&=\frac{F(\theta)}{\sqrt{3}}\left[\cosh\left(\frac{\theta+4\pi \mathrm{i}}{3}\right)-1\right].
\end{aligned}\tag{2.21}$$

Inserting (2.21) in (2.17) yields

$$\begin{aligned}
\phi(0,0\,|\,\theta)&=-\mathrm{i}F(\theta)\left[\sinh\left(\frac{\theta+\mathrm{i}\pi}{3}\right)-\mathrm{i}\sqrt{3}\right]\\
\phi(0,+\,|\,\theta)=\phi(+,0\,|\,\mathrm{i}\pi-\theta)&=\mathrm{i}F(\theta)\sinh\left(\frac{\theta}{3}\right)\\
\phi(+,+\,|\,\theta)&=-\mathrm{i}F(\theta)\left[\sinh\left(\frac{\theta+\mathrm{i}\pi}{3}\right)-\mathrm{i}\frac{\sqrt{3}}{2}\right]
\end{aligned}\tag{2.22}$$

where $F(\theta)=F(\mathrm{i}\pi-\theta)$. The rest of the amplitudes $\phi(\sigma, K\,|\,\theta)$ follows through the symmetries (2.9)-(2.12), that is

$$\phi(\sigma, K\,|\,\theta)=\phi(\sigma,-K\,|\,\theta)=\phi(-\sigma,-K\,|\,\theta).\tag{2.23}$$

In addition, crossing invariance holds

$$\phi(\sigma, K\,|\,\theta)=\phi(K,\sigma\,|\,\mathrm{i}\pi-\theta).\tag{2.24}$$

One can check that the YB relations hold for our S-matrix. It is here a consequence of the self-consistency of the Q, $\bar{Q}$ algebra.

Unitarity of the S-matrix imposes on the normalization factor $F(\theta)$:

$$F(\theta)F(-\theta)=\frac{4}{3}\frac{\cosh^2\theta/6}{\cosh^2\theta/2}.\tag{2.25}$$

In order to find $F(\theta)$, it is convenient to write it as

$$F(\theta)=\frac{2}{\sqrt{3}}[f(\theta)]^2$$

where

$$f(\theta)f(-\theta)=\frac{\cosh\theta/6}{\cosh\theta/2}.\tag{2.26}$$

Using the identity

$$\Gamma\left(\frac{1}{2}+\mathrm{i}\frac{z}{\pi}\right)\Gamma\left(\frac{1}{2}-\mathrm{i}\frac{z}{\pi}\right)=\frac{\pi}{\cosh z}\tag{2.27}$$

(2.26) can be written as

$$f(\theta)f(-\theta)=\phi(\theta)\phi(-\theta)\tag{2.28}$$

where

$$\phi(\theta)=\Gamma\left(\frac{1}{2}+\frac{\theta}{2\pi \mathrm{i}}\right)\Big/\Gamma\left(\frac{1}{2}+\frac{\theta}{6\pi \mathrm{i}}\right) \tag{2.29}$$

A minimal solution of equations (2.26) is given by the infinite product

$$f(\theta)=\prod_{l=0}^{\infty}\frac{\phi(\theta+2\pi l\mathrm{i})\phi((2l+1)\mathrm{i}\pi-\theta)\phi(2(l+1)\pi\mathrm{i})}{\phi(\theta+(2l+1)\mathrm{i}\pi)\phi(2(l+1)\pi\mathrm{i}-\theta)\phi(2l\pi\mathrm{i})} \tag{2.30}$$

That is

$$f(\theta)=\prod_{l=0}^{\infty}\frac{\Gamma(\frac{1}{2}+z+l)\Gamma(1-z+l)\Gamma\left(\frac{z+l+2}{3}\right)}{\Gamma(\frac{3}{2}-z+l)\Gamma(1+z+l)\Gamma\left(\frac{l+2-z}{3}\right)}\frac{\Gamma\left(\frac{1}{2}+\frac{1+l-z}{3}\right)\Gamma\left(\frac{1}{2}+\frac{l}{3}\right)}{\Gamma\left(\frac{1}{2}+\frac{l+z}{3}\right)\Gamma\left(\frac{1}{2}+\frac{l+1}{3}\right)}\left(l+\frac{1}{2}\right) \tag{2.31}$$

where $z=\theta/(2\pi\mathrm{i})$.

It is easy to check that this infinite product converges and it is analytic and without zeros on the physical strip.

Up to now we set the spin of Q and $\bar{Q}$ to be $+\frac{1}{3}$ and $-\frac{1}{3}$, respectively. In general we can consider the algebra (2.1) for non-local charges with spin $\pm s/3$, where s is not a multiple of 3. This follows from the previous construction by rescaling $\theta\to s\theta$ and $\omega\to\omega^s$. The representation of Q and $\bar{Q}$ on asymptotic states will depend on whether s is even or odd. We know that $Q^3=P_s$, is a local integral of motion with spin s, $\mathscr{C}$ parity $(-1)^{s+1}$ and Z_3-invariant. Therefore we need a new quantum number ε (odd under $\mathscr{C}$) when s is even. In this way $Q^3=P_s$ will change sign under $\mathscr{C}$.

Further S-matrices can be obtained by rescaling $\theta\to 2\theta$, $\omega\to\omega^2=\omega^*$. (That is taking the complex conjugate of the amplitude forms.) This changes the spin of Q^3 and $\bar{Q}^3$ to $s=2$ making them odd under $\mathscr{C}$.

Therefore, we introduce an extra quantum number $\varepsilon=\pm 1$. That is, the one particle states are now $|\varepsilon,\sigma,\theta\rangle$. The operator Q acts on them as follows

$$Q|\varepsilon,\sigma,\theta\rangle=\varepsilon\lambda_\theta\omega^\sigma|\varepsilon,\sigma+1,\theta\rangle \tag{2.32}$$

$$\begin{aligned}Q|\varepsilon_1,\sigma_1,\theta_1;\varepsilon_2,\sigma_2,\theta_2\rangle\\ &=\varepsilon_1\lambda_{\theta_1}\omega^{\sigma_1}|\varepsilon_1,\sigma_1+1,\theta_1;\varepsilon_2,\sigma_2,\theta_2\rangle\\ &\quad+\varepsilon_2\lambda_{\theta_2}\omega^{\sigma_2-\sigma_1}|\varepsilon_1,\sigma_1,\theta_1;\varepsilon_2,\sigma_2+1,\theta_2\rangle\end{aligned} \tag{2.33}$$

and analogous formulae hold for $\bar{Q}$.

The Z_3 and ε-conservation and the invariance under Q and $\bar{Q}$ now constrain the S-matrix to have the form

$$S^{\varepsilon_1,\sigma_1,\varepsilon_2,\sigma_2}_{\varepsilon_3,\sigma_3,\varepsilon_4,\sigma_4}(\theta)=\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4}\delta_{\varepsilon_1\varepsilon_3}\delta_{\varepsilon_2\varepsilon_4}\phi(\sigma_1+\sigma_2,\sigma_1-\sigma_3,\varepsilon_1\varepsilon_2|\theta). \tag{2.34}$$

We have two sectors: one with $\varepsilon_1\varepsilon_2=+1$ and another with $\varepsilon_1\varepsilon_2=-1$. They are connected by crossing since particles and antiparticles have opposite values of ε.

In the $\varepsilon_1\varepsilon_2=+1$ sector we set

$$\phi(\sigma,K,+|\theta)=\hat{\phi}(\sigma,K|-2\theta)F_+(\theta) \tag{2.35}$$

where the $\hat{\phi}(\sigma,K|2\theta)\equiv\phi(\sigma,K,\theta)/F(\theta)$ (see equations (2.22)) and $F_+(\theta)$ is a normalization factor.

In the $\varepsilon_1\varepsilon_2=-1$ sector we take

$$\phi(\sigma, K, -|\theta)=\hat{\phi}(\sigma, K\,|\,3\pi\mathrm{i}-2\theta)F_-(\theta). \tag{2.36}$$

The amplitudes (2.35)-(2.36) are invariant under Q and $\bar{Q}$ and under crossing symmetry:

$$\phi(\sigma, K, +|\,\theta)=\phi(K, \sigma, -|\,\mathrm{i}\pi-\theta). \tag{2.37}$$

The resulting amplitudes can be summarized as follows

$$\begin{aligned}
\phi(0,0,\varepsilon\,|\,\theta)&=\mathrm{i}F_\varepsilon(\theta)\left[\varepsilon\sinh\left(\frac{2\theta-\mathrm{i}\pi}{3}\right)+\mathrm{i}\sqrt{3}\right]\\
\phi(0,+,\varepsilon\,|\,\theta)&=\phi(+,0,\varepsilon\,|\,\mathrm{i}\pi-\theta)=-\mathrm{i}\varepsilon F_\varepsilon(\theta)\sinh\left(\frac{2\theta}{3}\right)\\
\phi(+,+,\varepsilon\,|\,\theta)&=\mathrm{i}\varepsilon F_\varepsilon(\theta)\left[\varepsilon\sinh\left(\frac{2\theta-\mathrm{i}\pi}{3}\right)+\mathrm{i}\frac{\sqrt{3}}{2}\right]
\end{aligned} \tag{2.38}$$

where $F_+(\theta)=F_-(\mathrm{i}\pi-\theta)$ are normalization factors. Unitarity constrains them to fulfil

$$\begin{aligned}
F_+(\theta)F_+(-\theta)&=\frac{4}{3}\frac{\cosh^2\theta/3}{\cosh^2\theta}\\
F_-(\theta)F_-(-\theta)&=\frac{4}{3}\frac{\sinh^2\theta/3}{\sinh^2\theta}.
\end{aligned} \tag{2.39}$$

Proceeding as before (cf (2.25)), we set

$$F_+(\theta)=\frac{2}{\sqrt{3}}[g(\theta)]^2 \qquad F_-(\theta)=\frac{2}{\sqrt{3}}[g(\mathrm{i}\pi-\theta)]^2.$$

Then equations (2.39) yield

$$g(\theta)g(-\theta)=\phi_1(\theta)\phi_1(-\theta),\ g(\mathrm{i}\pi-\theta)g(\mathrm{i}\pi+\theta)=\phi_2(\theta)\phi_2(-\theta) \tag{2.40}$$

where

$$\begin{aligned}
\phi_1(\theta)&=\Gamma\left(\frac{1}{6}+\frac{\theta}{3\pi\mathrm{i}}\right)\Gamma\left(\frac{5}{6}+\frac{\theta}{3\pi\mathrm{i}}\right)\\
\phi_2(\theta)&=\Gamma\left(\frac{1}{3}+\frac{\theta}{3\pi\mathrm{i}}\right)\Gamma\left(\frac{2}{3}+\frac{\theta}{3\pi\mathrm{i}}\right).
\end{aligned} \tag{2.41}$$

The minimal solution of equations (2.40) can be written as the infinite product

$$\begin{aligned}
g(\theta)=\prod_{l=0}^{\infty}&\frac{\Gamma\left(\frac{1}{6}+\frac{2l}{3}+z\right)\Gamma\left(\frac{5}{6}+\frac{2l}{3}+z\right)}{\Gamma\left(1+\frac{2l}{3}+z\right)\Gamma\left(\frac{2}{3}(l+1)+z\right)}\\
&\times\frac{\Gamma\left(\frac{2}{3}(l+1)-z\right)\Gamma\left(1+\frac{2l}{3}-z\right)\Gamma\left(\frac{3}{2}+\frac{2l}{3}\right)}{\Gamma\left(\frac{5}{6}+\frac{2l}{3}-z\right)\Gamma\left(\frac{3}{2}+\frac{2l}{3}-z\right)\Gamma\left(\frac{1}{6}+\frac{2l}{3}\right)}
\end{aligned} \tag{2.42}$$

where $z=\theta/(3\pi\mathrm{i})$.

3. Z_N charges, their coproduct and the associated S-matrices

We generalize in this section the construction of section 2 for Z_3 to Z_N for odd N. We consider two Z_N conserved charges Q and $\bar{Q}$ fulfilling the algebra

$$Q\bar{Q}=\omega\bar{Q}Q \qquad Q^N=P \qquad \bar{Q}^N=\bar{P} \tag{3.1}$$

where $\omega=\mathrm{e}^{2\pi i/N}$ is a Nth root for unity. We use particle states $|\sigma,\theta\rangle$ where the representation of Z_N is diagonal. Q and $\bar{Q}$ act on one-particle states as follows

$$\begin{aligned} Q|\sigma,\theta\rangle &= \lambda_\theta\omega^{d\sigma}|\sigma+1,\theta\rangle \\ \bar{Q}|\sigma,\theta\rangle &= \bar{\lambda}_\theta\omega^{-(d+1)\sigma}|\sigma-1,\theta\rangle \end{aligned} \tag{3.2a}$$

where $\sigma=0,1,2,\ldots,N-1 \bmod N$, d is an integer to be determined below and

$$\lambda_\theta=(m\,\mathrm{e}^{\theta})^{1/N} \qquad \bar{\lambda}_\theta=(m\,\mathrm{e}^{-\theta})^{1/N}. \tag{3.2b}$$

That is, Q and $\bar{Q}$ have (Lorentz) spin $1/N$. Representations of Q and $\bar{Q}$ having spin s/N will be obtained below by changing $\omega\to\omega^s$, $\theta\to s\theta$†.

As is easy to check, equations (3.1) hold for the one-particle representation (3.2). We define the action of Q and $\bar{Q}$ on two-particle states as follows

$$\begin{aligned} Q|\sigma_1\theta_1,\sigma_2\theta_2\rangle &= \lambda_1\omega^{d\sigma_1}|\sigma_1+1,\theta_1,\sigma_2,\theta_2\rangle+\lambda_2\omega^{d\sigma_2-\sigma_1}|\sigma_1,\theta_1,\sigma_2+1,\theta_2\rangle \\ \bar{Q}|\sigma_1\theta_1,\sigma_2\theta_2\rangle &= \bar{\lambda}_1\omega^{-(d+1)\sigma_1}|\sigma_1-1,\theta_1,\sigma_2\theta_2\rangle+\bar{\lambda}_2\omega^{-(d+1)\sigma_2-\sigma_1}|\sigma_1,\theta_1,\sigma_2-1,\theta_2\rangle. \end{aligned} \tag{3.3}$$

We check from (3.2) and (3.3) that equations (3.1) hold on one and two-particle states. For n-particle states this is also true provided one defines the action of Q and $\bar{Q}$ as

$$\begin{aligned} Q|\underset{\sim}{\sigma},\underset{\sim}{\theta}\rangle &= \sum_{i=1}^{n}\lambda(\theta_i)\omega^{d\sigma_i-\sum_{j=1}^{i-1}\sigma_j}|\underset{\sim}{\sigma}^{(i)},\underset{\sim}{\theta}\rangle \\ \bar{Q}|\underset{\sim}{\sigma},\underset{\sim}{\theta}\rangle &= \sum_{i=1}^{n}\bar{\lambda}(\theta_i)\omega^{-(d+1)\sigma_i-\sum_{j=1}^{i-1}\sigma_j}|\underset{\sim}{\bar{\sigma}}^{(i)},\underset{\sim}{\theta}\rangle \end{aligned} \tag{3.4}$$

where d is an integer

$$\begin{aligned} \sigma^{(i)} &\equiv (\sigma_1,\ldots,\sigma_{i-1},\sigma_i+1,\sigma_{i+1},\ldots,\sigma_n) \\ \bar{\sigma}^{(i)} &\equiv (\sigma_1,\ldots,\sigma_{i-1},\sigma_i-1,\sigma_{i+1},\ldots,\sigma_n). \end{aligned} \tag{3.5}$$

As, in the Z_3-case (section 2) the two-body S-matrix will have the Z_N-invariant form

$$S^{\sigma_1,\sigma_2}_{\sigma_1-K,\sigma_2+K}(\theta)=\alpha(\sigma_1,\sigma_2,K\,|\,\theta). \tag{3.6}$$

We now require Q and $\bar{Q}$ to be conserved. Therefore, they will have identical form on in and out states. These conservations imply on the S-matrix the constraints

$$\begin{aligned} &\alpha(\sigma_1,\sigma_2,K+1,\theta)\omega^{-(d+1)(\sigma_1+1+K)}-\omega^{-(d+1)\sigma_2-\sigma_1}\alpha(\alpha_2,\sigma_2-1,K,\theta) \\ &\quad=\lambda^2[\omega^{-(d+1)\sigma_1}\alpha(\sigma_1-1,\sigma_2,K+1,\theta) \\ &\qquad-\omega^{-(d+1)(\sigma_2-K)-\sigma_1-K}\alpha(\sigma_1,\sigma_2,K,\theta)] \end{aligned} \tag{3.7}$$

† It has recently been shown that there is no realization of (1.1) with $s=1$ and $N>5$ in perturbed unitary conformal invariant models [11]. The connection between our $s=1$ construction and [11] is to be investigated.

$$\lambda^2[\alpha(\sigma_1,\sigma_2,K,\theta)-\omega^{d(\sigma_2-K)-(d+1)\sigma_1}\alpha(\sigma_1,\sigma_2+1,K+1,\theta)]$$
$$=\omega^{-dK}\alpha(\sigma_1+1,\sigma_2,K)-\omega^{-(d+1)(\sigma_1+K)-1+d(\sigma_2-K-1)}$$
$$\times\,\alpha(\sigma_1,\sigma_2,K+1,\theta) \tag{3.8}$$

where $\lambda=\sqrt{\lambda_1/\lambda_2}=\mathrm{e}^{-\theta/2N}$.

For $N=3$ equations (3.7)-(3.8) become identical to (2.13)-(2.14). Equations (3.7)-(3.8) admit a simple PT-invariant ansatz analogous to (2.15) when

$$2d+1=N. \tag{3.9}$$

This means, in particular, that N must be odd, as we shall assume from now on, for simplicity. Then, we can set as in the Z_3 case (2.15)

$$\alpha(\sigma_1,\sigma_2,K,\theta)=\phi(\sigma_1+\sigma_2,K,\theta). \tag{3.10}$$

Inserting (3.10) in (3.7)-(3.8) yields

$$\phi(\sigma,K,\theta)-\omega^{-d(K-\sigma)}\phi(\sigma-1,K-1,\theta)$$
$$=\lambda_\theta^2[\omega^{-dK}\phi(\sigma-1,K,\theta)-\omega^{d(\sigma-1)}\phi(\sigma,K-1,\theta)] \tag{3.11}$$

and

$$\phi(\sigma,K,\theta)-\omega^{-d(K+\sigma)}\phi(\sigma+1,K-1,\theta)$$
$$=\lambda_\theta^2[\omega^{-dK}\phi(\sigma+1,K,\theta)-\omega^{-d(\sigma+1)}\phi(\sigma,K-1,\theta)]. \tag{3.12}$$

Equation (3.12) follows from (3.11) together with P and $\mathbb{C}$-invariance (see (2.10) and (2.11)).

It is convenient to introduce now the discrete Fourier transform

$$\phi(\sigma,K,\theta)=\omega^{d\sigma K}\sum_\alpha\omega^{\sigma\alpha}\psi(\alpha,K,\theta). \tag{3.13}$$

Inserting (3.13) in (3.11) yields the following recursion relation for $\psi(\alpha,K,\theta)$

$$\psi(\alpha,K,\theta)=\frac{\sin\left[\dfrac{\pi}{N}(K-\alpha-1)+\dfrac{\mathrm{i}\theta}{2N}\right]}{\sin\left[\dfrac{\pi}{N}(K-\alpha)-\dfrac{\mathrm{i}\theta}{2N}\right]}\psi(\alpha,K-1,\theta) \tag{3.14}$$

where we used

$$\lambda_\theta=\mathrm{e}^{-\theta/2N} \tag{3.15}$$

in accordance with (3.1).

We find from (3.14)

$$\psi(\alpha,K,\theta)=\prod_{m=-\alpha}^{K-1-\alpha}\frac{\sin\left[\dfrac{m\pi}{N}-\dfrac{\mathrm{i}\theta}{2N}\right]}{\sin\left[\dfrac{\pi(m+1)}{N}+\dfrac{\mathrm{i}\theta}{2N}\right]}\psi(\alpha,0,\theta). \tag{3.16}$$

This formula can be written in terms of the functions

$$X_m(\theta)=\prod_{k=0}^{m-1}\sin\left(\frac{2\pi k-\mathrm{i}\theta}{2N}\right)\prod_{k=m}^{d-1}\sin\left(\frac{2\pi(k+1)+\mathrm{i}\theta}{2N}\right)\qquad 0\leqslant m\leqslant d. \tag{3.17}$$

Notice that these functions are real for purely imaginary θ. They enjoy the following properties

$$X_m(\theta)=X_{-m}(\theta)=X_{N-m}(\theta) \tag{3.18}$$

$$X_0(\theta)\equiv w(\theta)=\prod_{k=0}^{d-1}\sin\left(\frac{2\pi(k+1)+\mathrm{i}\theta}{2N}\right) \tag{3.19}$$

and

$$w(\theta)w(-\theta)=\frac{1}{2^{N-1}}\frac{\sinh(\theta/2)}{\sinh(\theta/2N)}. \tag{3.20}$$

Using (3.17) we can write (3.16) as

$$\psi(\alpha, K, \theta)=\frac{X_{K-\alpha}(\theta)}{X_\alpha(\theta)}\,\psi(\alpha,0,\theta). \tag{3.21}$$

Let us now enforce P or PT-invariance in order to fix $\psi(\alpha,0,\theta)$. Equations (2.12a) and (3.13) yield, as the P-invariance constraint on $\psi(\alpha, K, \theta)$,

$$\psi(\alpha, K, \theta)=\psi(\alpha-K, -K, \theta). \tag{3.22}$$

The solution (3.21) satisfies (3.22) provided

$$\psi(\alpha,0,\theta)=F_N(\theta)[X_\alpha(\theta)]^2 \tag{3.23}$$

where $F_N(\theta)$ is an arbitrary function of θ. Therefore, the solution (3.21) takes the form

$$\psi(\alpha, K, \theta)=F_N(\theta)X_\alpha(\theta)X_{K-\alpha}(\theta). \tag{3.24}$$

Let us now investigate the crossing symmetry of these amplitudes $\psi(\alpha, K, \theta)$. Crossing invariance (2.12) yields

$$\sum_{\alpha=0}^{N-1}\omega^{\sigma\alpha}\psi(\alpha, K, \mathrm{i}\pi-\theta)=\sum_{\alpha=0}^{N-1}\omega^{K\alpha}\psi(\alpha,\sigma,\theta). \tag{3.25}$$

It is then convenient to define an additional finite Fourier transform

$$X(\alpha,\beta,\theta)=\sum_{K=0}^{N-1}\omega^{-\beta K}\psi(\alpha, K, \theta). \tag{3.26}$$

Crossing thus requires

$$X(\alpha,\beta,\theta)=X(\beta,\alpha,\mathrm{i}\pi-\theta). \tag{3.27}$$

Inserting now (3.24) in (3.26) yields

$$X(\alpha,\beta,\theta)=F_N(\theta)X_\alpha(\theta)\sum_{K=0}^{N-1}\omega^{-\beta K}X_{K-\alpha}(\theta). \tag{3.28}$$

We can evaluate this sum with the help of the self-duality property [10]

$$\sum_{k=0}^{N-1}\omega^{kl}X_k(\theta)=\frac{X_l(\mathrm{i}\pi-\theta)}{X_0(\mathrm{i}\pi-\theta)}\sum_{k=0}^{N-1}X_k(\theta). \tag{3.29}$$

Moreover, the sum in the RHS can be computed with the result

$$\sum_{K=0}^{N-1}X_K(\theta)=\sqrt{N}\prod_{l=0}^{d-1}\sin\left[\frac{\pi(2l+1)-\mathrm{i}\theta}{2N}\right]=\sqrt{N}\,X_0(\mathrm{i}\pi-\theta). \tag{3.30}$$

Then, we find

$$\sum_{k=0}^{N-1} \omega^{km} X_k(\theta) = \sqrt{N} X_m(\mathrm{i}\pi - \theta) \tag{3.31}$$

and

$$X(\alpha, \beta, \theta) = F_N(\theta)\sqrt{N}\omega^{-\alpha\beta} X_\alpha(\theta) X_\beta(\mathrm{i}\pi - \theta). \tag{3.32}$$

We recall that the weights in the Z_N-invariant model of [10] are expressed as Z_N-Fourier transforms of the functions $X_\alpha(\theta)$. We find in the present case statistical weights that are double Z_N-Fourier transforms of (3.32), bilinear in the functions $X_\alpha(\theta)$.

In order to impose crossing (equation (3.27)), we require $F_N(\theta)$ to be a crossing symmetric function

$$F_N(\theta) = F_N(\mathrm{i}\pi - \theta) \tag{3.33}$$

to be determined by imposing unitarity and arbitrary. In conclusion, the C, T, P and crossing invariant S-matrix can be written as

$$\psi(\alpha, K, \theta) = F_N(\theta) X_\alpha(\theta) X_{K-\alpha}(\theta) \tag{3.34}$$

and (cf (3.13))

$$\phi(\sigma, K, \theta) = \omega^{d\sigma K} F_N(\theta) \sum_{\alpha=0}^{N-1} \omega^{\sigma\alpha} X_\alpha(\theta) X_{K-\alpha}(\theta) \tag{3.35}$$

where $F_N(\theta)$ fulfils (3.33).

It is easy to check real analyticity for $\phi(\sigma, K, \theta)$ from (3.35). That is

$$\phi(\sigma, K, \theta)^* = \phi(\sigma, K, -\theta^*) \tag{3.36}$$

provided $F_N(\theta)^* = F_N(-\theta^*)$.

One can check that (3.35) fulfils the Yang–Baxter equations.

Let us now consider the unitarity property of the S-matrix (3.35). It takes the form

$$\sum_K \phi(\sigma, K, \theta)\phi(\sigma', K, -\theta) = \delta_{\sigma\sigma'}. \tag{3.37}$$

We find from (3.35) setting $\sigma = \sigma'$

$$F_N(\theta)F_N(-\theta) = N\left[\sum_{\alpha=0}^{N-1} X_\alpha(\theta) X_\alpha(-\theta)\right]^{-2}. \tag{3.38}$$

The sum in (3.38) can easily be computed using

$$X_m(\theta)X_m(-\theta) = \frac{1}{2^{N-1}} \frac{\sinh(\theta/2N)\sinh(\theta/2)}{\sin^2(m\pi/N) + \sinh^2(\theta/2N)} \tag{3.39}$$

that generalizes (3.20).

We find after calculation

$$\sum_{\alpha=0}^{N-1} X_\alpha(\theta) X_\alpha(-\theta) = \frac{N}{2^{N-1}} \frac{\cosh(\theta/2)}{\cosh(\theta/2N)}. \tag{3.40}$$

Using now equations (3.38) and (3.40) yields

$$F_N(\theta)F_N(-\theta) = \left(\frac{2^{N-1}\cosh(\theta/2N)}{\cosh(\theta/2)}\right)^2 \frac{1}{N}. \tag{3.41}$$

It must be noticed that the identification of θ with the physical rapidity (and hence the crossing transformation $\theta \to \mathrm{i}\pi - \theta$) is not unique. We can rescale $\omega \to \omega^s$ and $\theta \to s\theta$ where the integer s is not a divisor of N. This changes the spin of Q and $\bar{Q}$ to s/N.

We shall discuss two cases in more detail: $s = N-2$ and $s = N-1$. Let us start by $s = N-2$. We call this case model I. We can write the S-matrix from equations (3.6), (3.10) and (3.35)

$$S^{\sigma-M,M}_{\sigma-M-K,M+K}(\theta) = \tilde{\phi}(\sigma, K, \theta) = \omega^{\sigma K} \tilde{F}_N(\theta) \sum_{\alpha=0}^{N-1} \omega^{-2\sigma\alpha} \tilde{X}_\alpha(\theta) \tilde{X}_{K-\alpha}(\theta) \tag{3.42}$$

where

$$\tilde{X}_\alpha(\theta) = \prod_{k=0}^{\alpha-1} \sin\left(\frac{2\pi k}{N} + \frac{\mathrm{i}(N-2)}{2N}\theta\right) \prod_{k=\alpha}^{d-1} \sin\left[\frac{2\pi}{N}(k+1) - \frac{\mathrm{i}\theta}{2N}(N-2)\right]. \tag{3.43}$$

The unitarization function $\tilde{F}_N(\theta)$ obeys here

$$\tilde{F}_N(\theta)\tilde{F}_N(-\theta) = \frac{1}{N}\left[2^{N-1} \frac{\cosh\left(\dfrac{N-2}{2N}\theta\right)}{\cosh\left(\dfrac{N-2}{2}\theta\right)}\right]^2 \tag{3.44}$$

$$\tilde{F}_N(\theta) = \tilde{F}_N(\mathrm{i}\pi - \theta). \tag{3.45}$$

Notice that $\tilde{F}_N(\theta)$ cannot be obtained by rescaling θ in $F_N(\theta)$ since crossing always contains an $\mathrm{i}\pi$ shift [cf (3.33), (3.41), (3.44) and (3.45)].

In order to solve the functional equations (3.44) and (3.45) it is convenient to write

$$\tilde{F}_N(\theta) = \frac{2^{N-1}}{\sqrt{N}} [\tilde{f}_N(\theta)]^2 B_N(\theta) \tag{3.46}$$

where

$$\tilde{f}_N(\theta)\tilde{f}_N(-\theta) = \frac{\cosh\left(\dfrac{N-2}{2N}\theta\right)}{\cosh\left(\dfrac{N-2}{2}\theta\right)} \quad \text{and} \quad \tilde{f}_N(\theta) = \tilde{f}_N(\mathrm{i}\pi - \theta). \tag{3.47}$$

Using the identity (2.27), we can write (3.47) as

$$\tilde{f}_N(\theta)\tilde{f}_N(-\theta) = \phi(\theta)\phi(-\theta) \tag{3.48}$$

where

$$\phi(\theta) = \frac{\Gamma\left(\dfrac{1}{2} + \dfrac{(N-2)\theta}{2\pi\mathrm{i}}\right)}{\Gamma\left(\dfrac{1}{2} + \dfrac{(N-2)\theta}{2N\pi\mathrm{i}}\right)} \tag{3.49}$$

is analytic for $\operatorname{Im}\theta > -\pi/(N-2)$.

In an analogous way to section 2, an infinite product solution for $\tilde{f}_N(\theta)$ can be obtained with the result

$$B_N(\theta) = \prod_{m=1}^{(N-3)/2} \frac{\sinh\left(\dfrac{\theta}{2} + \dfrac{\mathrm{i}m\pi}{N-2}\right)}{\sinh\left(\dfrac{\theta}{2} - \dfrac{\mathrm{i}m\pi}{N-2}\right)} \tag{3.50}$$

$$\tilde{f}_N(\theta)=\prod_{m=0}^{\infty}\frac{\Gamma\left(z+\frac{1}{2}+(N-2)m\right)\Gamma\left(\frac{z}{N}+1-\frac{1}{N}+\frac{N-2}{N}m\right)\Gamma\left(-z+\frac{N-1}{2}+(N-2)m\right)}{\Gamma\left(\frac{z}{N}+\frac{1}{2}+\frac{N-2}{N}m\right)\Gamma\left(z+\frac{N-1}{2}+(N-2)m\right)\Gamma\left(-\frac{z}{N}+1-\frac{1}{N}+\frac{N-2}{N}m\right)}$$
$$\times\frac{\Gamma\left(-\frac{z}{N}+\frac{1}{2}+\frac{N-2}{N}(m+1)\right)\Gamma\left(\frac{1}{2}+(N-2)(m+1)\right)\Gamma\left(\frac{1}{2}+\frac{N-2}{N}m\right)}{\Gamma\left(-z+\frac{1}{2}+(N-2)(m+1)\right)\Gamma\left(\frac{1}{2}+\frac{N-2}{N}(m+1)\right)\Gamma\left(\frac{1}{2}+(N-2)m\right)} \tag{3.51}$$

where $z\equiv(N-2)\theta/(2\pi\mathrm{i})$.

In (3.51) the infinite product of Γ-functions has no zero neither poles in the physical strip. Poles and zeros are on the pole factor (the sinh functions). We choose this pole factor $B_N(\theta)$ to be one associated with the bound-state mass spectrum

$$m_k=2m\sin\left(\frac{\pi k}{N-2}\right)\qquad 1\leqslant k\leqslant\frac{N-3}{2}. \tag{3.52}$$

These bound states plus the mass m particle coincides with the $B_{N-1/2}$ spectrum. The interpretation of this S-matrix as describing a perturbed CFT is given in the next section.

Let us now discuss the S-matrix obtained from (3.35) through the rescaling

$$\theta\to(N-1)\theta\qquad \omega\to\omega^{N-1}=\omega^{-1}=\omega^* \tag{3.53}$$

We call this case, model II.

For model II we add an extra quantum number $\varepsilon=\pm$ to characterize the particle states. As in section 2, the reason for introducing is to allow Q^N (a Z_N-invariant operator with even spin) to be odd under charge conjugation.

Thus, our asymptotic states will now be

$$|\varepsilon_1\sigma_1\theta_1,\varepsilon_2\sigma_2\theta_2,\ldots,\varepsilon_N\sigma_N\theta_N\rangle\qquad \varepsilon_i=\pm1. \tag{3.54}$$

The Z_N charges, Q and $\bar{Q}$ are diagonal on the ε indices and act as follows on one and two particle states

$$Q|\varepsilon\sigma\theta\rangle=\varepsilon\lambda_\theta\omega^{d\sigma}|\varepsilon,\sigma+1,\theta\rangle$$

$$\begin{aligned}Q|\varepsilon_1\sigma_1\theta_1;\varepsilon_2\sigma_2\theta_2\rangle&\\ &=\varepsilon_1\lambda_{\theta_1}\omega^{d\sigma_1}|\varepsilon_1,\sigma_1+1,\theta_1,\varepsilon_2\sigma_2\theta_2\rangle\\ &\quad+\varepsilon_2\lambda_{\theta_2}\omega^{d\sigma_2-\sigma_1}|\varepsilon_1\sigma_1\theta_1;\varepsilon_2,\sigma_2+1,\theta_2\rangle.\end{aligned} \tag{3.55}$$

Analogous formulae hold for $\bar{Q}$ and for states with more particles.

The conservation of Q and $\bar{Q}$ constrains the two-particle S-matrix to have the form

$$S^{\varepsilon_1\sigma_1,\varepsilon_2\sigma_2}_{\varepsilon_3\sigma_3,\varepsilon_4\sigma_4}(\theta)=\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4}\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\Phi(\sigma_1+\sigma_2|,\sigma_1-\sigma_3,\varepsilon_1\varepsilon_2,\theta). \tag{3.56}$$

We have two separate sectors depending whether $\varepsilon_1\varepsilon_2=2+1$ or -1. In the $\varepsilon_1\varepsilon_2=+1$ sector we set

$$\Phi(\sigma,K,+,\theta)=\hat{\phi}(\sigma,K,\theta)\hat{F}_N(\theta) \tag{3.57}$$

where

$$\hat{\phi}(\sigma, K, \theta) = \omega^{-d\sigma K} \sum_{\alpha=0}^{N-1} \omega^{-\sigma\alpha} X_\alpha((1-N)\theta) X_{K-\alpha}((1-N)\theta) \tag{3.58}$$

and the functions $X_\alpha(\phi)$ are given by (3.17).

In the sector $\varepsilon_1\varepsilon_2 = -1$, we set

$$\Phi(\sigma, K, -, \theta) = \hat{F}_N(\mathrm{i}\pi - \theta)\hat{\phi}\left(\sigma, K, \theta - \frac{\mathrm{i}\pi N}{N-1}\right). \tag{3.59}$$

It can be checked that equations (3.57)–(3.59) are invariant under Q and $\bar{Q}$.

Crossing invariance requires here that

$$\Phi(\sigma, K, \varepsilon, \theta) = \Phi(K, \sigma, -\varepsilon, \mathrm{i}\pi - \theta). \tag{3.60}$$

Using now equations (3.23), (3.24) and (3.56) we find that

$$\hat{\phi}(\sigma, K, \mathrm{i}\pi - \theta) = \hat{\phi}\left(K, \sigma, \theta - \frac{\mathrm{i}\pi N}{N-1}\right). \tag{3.61}$$

This, together with equations (3.57) and (3.59) shows that crossing invariance (3.60) holds

Let us finally find the unitarization factor $\hat{F}_N(\theta)$. We find two equations from the sector $\varepsilon_1\varepsilon_2 = +1$ and $\varepsilon_1\varepsilon_2 = -1$, (respectively:

$$\begin{aligned} \hat{F}_N(\theta)\hat{F}_N(-\theta) &= \frac{1}{N}\left[\frac{2^{N-1}\cosh\left(\frac{N-1}{2N}\theta\right)}{\cosh\left(\frac{N-1}{2}\theta\right)}\right]^2 \\ \hat{F}_N(\mathrm{i}\pi - \theta)\hat{F}_N(\mathrm{i}\pi + \theta) &= \frac{1}{N}\left[\frac{2^{N-1}\sinh\left(\frac{N-1}{2N}\theta\right)}{\sinh\left(\frac{N-1}{2}\theta\right)}\right]^2. \end{aligned} \tag{3.62}$$

Proceeding as earlier ((3.46)–(3.52)), we set

$$\hat{F}_N(\theta) = \frac{2^{N-1}}{\sqrt{N}}[\hat{f}_N(\theta)]^2 D_N(\theta). \tag{3.63}$$

We have then from (3.62) after some calculation

$$\begin{aligned} \hat{f}_N(\theta)\hat{f}_N(-\theta) &= \phi_1(\theta)\phi_1(-\theta) \\ \hat{f}_N(\mathrm{i}\pi - \theta)\hat{f}_N(\mathrm{i}\pi + \theta) &= \phi_2(\theta)\phi_2(-\theta) \end{aligned} \tag{3.64}$$

where

$$\phi_1(\theta) = \frac{\Gamma(\frac{1}{2} + Nz)}{\Gamma(\frac{1}{2} + z)} \qquad \phi_2(\theta) = \sqrt{N}\frac{\Gamma(Nz)}{\Gamma(z)} \tag{3.65}$$

and

$$z \equiv \frac{N-1}{2N\pi\mathrm{i}}\theta.$$

The solution of equations (3.64) can be expressed as the following infinite product:

$$D_N(\theta)=\prod_{l=1}^{N-2}\frac{\sinh\left(\frac{\theta}{2}+\frac{il\pi}{2(N-1)}\right)}{\sinh\left(\frac{\theta}{2}-\frac{il\pi}{2(N-1)}\right)}$$

$$\hat{f}_N(\theta)=\prod_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}+Nz+(N-1)k)\Gamma(-Nz+(k+\frac{1}{2})(N-1))\Gamma(\frac{1}{2}+(N-1)(k+1))}{\Gamma(\frac{1}{2}-Nz+(N-1)(k+1))\Gamma(Nz+(k+\frac{1}{2})(N-1))\Gamma(\frac{1}{2}+(N-1)k)}$$
$$\times\frac{\Gamma\left(z+\frac{N-1}{N}\left(k+\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}-z+\frac{N-1}{N}(k+1)\right)\Gamma\left(\frac{1}{2}+\frac{N-1}{N}k\right)}{\Gamma\left(\frac{1}{2}+z+\frac{N-1}{N}k\right)\Gamma\left(-z+\frac{N-1}{N}\left(k+\frac{1}{2}\right)\right)\Gamma\left(\frac{1}{2}+\frac{N-1}{N}(k+1)\right)}. \tag{3.66}$$

We introduced as pole factor the one associated to a D_N-spectrum. That is, model II contains, besides the particle of mass m, $N-2$ bound-states with masses

$$m_k=2m\sin\frac{\pi k}{2(N-1)}\qquad 1\leqslant k\leqslant N-2. \tag{3.67}$$

4. Perturbed parafermionic conformal field theories as integrable massive field theories

In this section we investigate integrable perturbations of parafermionic CFT leading to massive integrable field theories. Their associated S-matrices being found in section 3.

Let us start with model I. We claim that this QFT follows from two parafermionic Z_N models perturbed by the product of the thermal operators of both models. That is, a scaling model described by the Hamiltonian

$$\mathscr{H}^{\mathrm{I}}_{\lambda}=\mathscr{H}_0(Z_N)+\mathscr{H}_0(Z'_N)+\lambda\int \mathrm{d}^2x\,\varepsilon_1\varepsilon'_1 \tag{4.1}$$

where Z_N and Z'_N refers to the two parafermionic CFT and $\varepsilon_1(\varepsilon'_1)$ is the Z_N-neutral field in the $Z_N(Z'_N)$ model with conformal dimensions [9]

$$D_1=\overline{D_1}=\frac{2}{N+2}. \tag{4.2}$$

In equations (4.1) $\lambda\to 0$ is a (small) parameter. We find by dimensional analysis $\dim\lambda=2(N-2)/(N+2)$. Therefore, the correlation length behaves here as

$$\xi\sim\lambda^{-(N+2)/2(N-2)}. \tag{4.3}$$

Moreover, we identify the conserved Z_N-charge Q with the operator

$$Q=Q\int \mathrm{d}z\,\psi_1\psi'_1+\lambda\int \Phi\,\mathrm{d}\bar{z} \tag{4.4}$$

where $\psi_1(\psi'_1)$ is the parafermionic field of the model $Z_N(Z'_N)$ with conformal dimensions

$$\Delta_1=1-\frac{1}{N}\qquad \bar{\Delta}_1=0. \tag{4.5}$$

As is easy to check, the identification (4.4) is consistent with the value $S=1-2/N$ for the charge Q (see section 3).

In the massive theory (4.1) $\psi_1\psi_1'$ depends both on z and $\bar{z}$ and we assume it is the z component of a conserved current. We therefore set

$$\partial_{\bar{z}}[\psi_1\psi_1'] = \lambda\partial_z\Phi \tag{4.6}$$

where Φ should be a local operator. Dimensional counting yields from equations (4.3), (4.5) and (4.6)

$$\begin{aligned}\Delta_\Phi &= 2\frac{N-2}{N(N+2)} \\ \bar{\Delta}_\Phi &= 2\left(\frac{N-2}{N(N+2)}+\frac{1}{N}\right).\end{aligned} \tag{4.7}$$

These are precisely the conformal dimensions of the operator $\Phi^{(2)}_{[2,0]}(z,\bar{z})$ in the Z_N parafermionic CFT [9]. These operators $\Phi^{(k)}_{[q,\bar{q}]}$ are obtained from the other parameters $\sigma_k(z,\bar{z})$ by applying the operator $\bar{A}_\nu$ (see [9]). In our case:

$$\Phi^{(2)}_{[2,0]}(z,\bar{z}) = \bar{A}^{+}_{-1/N}\sigma_2(z,\bar{z}). \tag{4.8}$$

(We recall that the conformal dimension of σ_2 are $d_2=\bar{d}_2=(N-2)/N(N+2)$.) That is we set

$$\Phi_1(z,z') = C_1\Phi^{(2)}_{[2,0]}(z,\bar{z})\Phi^{(2)\prime}_{[2,0]}(z,\bar{z}) \tag{4.9}$$

where C_1 is some numerical constant. These arguments prove that our claim is consistent.

Let us now consider model II. We identify it with two parafermionic Z_{2N} CFT perturbed by the product of the thermal operator in both models. That is,

$$\mathcal{H}^{11}_\lambda = \mathcal{H}_0(Z_{2N}) + \mathcal{H}_0(Z'_{2N}) + \lambda\int \mathrm{d}^2x\,\varepsilon_1\varepsilon_1'. \tag{4.10}$$

Now, dimensional analysis yields $\dim\lambda = 2(N-1)/(N+1)$ and hence

$$\xi \sim \lambda^{-N+1/2(N-2)}. \tag{4.11}$$

We identify the Z_N-charge in model II with

$$Q = \int \mathrm{d}z\,\psi_1\psi_1' + \lambda\int \Phi\,\mathrm{d}\bar{z} \tag{4.12}$$

where now the Z_N and Z'_N parafermionic operator have conformal dimensions

$$\Delta_1 = 1-\frac{1}{2N} \qquad \bar{\Delta}_1 = 0. \tag{4.13}$$

This is consistent with the spin value $1-1/N$ for Q in model II.

We find as the conserved current in the $Z_{2N}\otimes Z'_{2N}$ perturbed model (4.10)

$$\partial_{\bar{z}}[\psi_1\psi_1'] = \lambda\partial_z\Phi_{11}$$

where

$$\Phi_{11}(z,\bar{z}) = C_{11}\Phi^{(2)}_{[2,0]}\Phi^{(2)\prime}_{[2,0]} \tag{4.14}$$

where C_{11} is a number and $\Phi^{(2)}_{[2,0]}(z,\bar{z})$ is the same operator as in (4.8) but now in the Z_{2N}-parafermionic theory.

In model II, where $s=N-1$, the local operator $P_s=Q^N$ can be represented in terms of operators of the Z_{2N} parafermionic CFT as

$$P_{N-1}=\int \mathrm{d}z\,\psi_N(z)\psi'_N(z)+\lambda K\int \mathrm{d}\bar{z}\,\Phi^{(2)}_{[2N,0]}\Phi^{(2)\prime}_{[2N,0]} \tag{4.15}$$

where K is a numerical constant and the operators $\Phi^{(2)}_{[2N,0]}$ defined in [9] have conformal dimensions

$$\Delta^2_{[2N,0]}=\frac{N(N-1)}{2(N+1)}\quad\text{and}\quad \bar{\Delta}^{(2)}_{[2N,0]}=\frac{1}{N+1}.$$

References

[1] Zamolodchikov A B 1988 *Int. J. Mod. Phys.* **A3** 743
[2] Ahn C, Bernard D and Le Clair A 1990 *Nucl. Phys.* B **346** 409
Bernard D and Le Clair C J 1990 *Phys. Lett.* **247B** 309
Fateev V A and Zamolodchikov A B 1990 *Int. J. Mod. Phys.* **A5** 1025
Fendley P, Mathur S D, Vafa C and Warnes N P 1990 *Phys. Lett.* **243B** 257
Smirnov F A 1991 *Int. J. Mod. Phys.* **A6** 1407
Sotkov G and Zhu C J 1989 *Phys. Lett.* **229B** 391
[3] Lüscher M 1978 *Nucl. Phys.* B **135** 9
[4] de Vega H J, Eichenherr H and Maillet J-M 1984 *Nucl. Phys.* B **240** 377
[5] Bernard D 1991 *Commun. Math. Phys.* **137** 191
C Destri and H J de Vega *Preprint PTHE-PAR-92-09*
[6] Fateev V A 1991 *Int. J. Mod. Phys.* **A6** 2109
[7] Destri C, de Vega H J and Fateev V A 1991 *Phys. Lett.* **256B** 173
[8] Zamolodchikov A B 1989 *Landau Institute preprint*
[9] Zamolodchikov A B and Fateev V A 1985 *Sov. Phys-JETP* **89** 380
[10] Fateev V A and Zamolodchikov A B 1982 *Phys. Lett.* A **92** 37
[11] Ravanini F *Saclay preprint SPhT/91-121*